• ES)

Invertire la seguente trasformata

$$X(z) = \frac{1}{(1-0{,}8z^{-1})(1{,}2+z^{-1})}$$

• lavoro in z^{-1} e decompongo in fratti semplici

$$X(z) = \frac{N(z)}{D(z)} = \frac{A_1}{(1-0{,}8z^{-1})} + \frac{A_2}{(1{,}2+z^{-1})} = \frac{A_1(1{,}2+z^{-1}) + A_2(1-0{,}8z^{-1})}{(1-0{,}8z^{-1})(1{,}2+z^{-1})}$$

$$N(z) = 1{,}2\,A_1 + A_1 z^{-1} + A_2 - 0{,}8\,A_2 z^{-1} = z^{-1}(A_1 - 0{,}8A_2) + (1{,}2A_1 + A_2) = 1$$

$$\begin{cases} A_1 - 0{,}8A_2 = 0 \\ 1{,}2A_1 + A_2 = 1 \end{cases} \quad \begin{cases} A_1 = 0{,}8\,A_2 \\ 0{,}96\,A_2 + A_2 = 1 \end{cases} \quad \begin{cases} A_1 = 0{,}8\,A_2 \\ A_2 = \frac{1}{1{,}96} \end{cases} \quad \begin{cases} A_1 = \frac{0{,}8}{1{,}96} \simeq 0{,}41 \\ A_2 = \frac{1}{1{,}96} \simeq 0{,}51 \end{cases}$$

$$X(z) = \frac{0{,}41}{1-0{,}8z^{-1}} + \frac{0{,}51}{1{,}2+z^{-1}} = \frac{0{,}41}{1-0{,}8z^{-1}} + \frac{0{,}51}{1{,}2\left(1+\frac{1}{1{,}2}z^{-1}\right)} = 0{,}41 \cdot \frac{z}{z-0{,}8} + \frac{0{,}51}{1{,}2}\cdot\frac{z}{z+\frac{1}{1{,}2}}$$

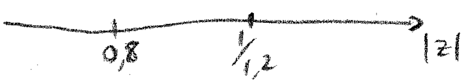
• linearità: trasformo i vari termini

$$0{,}41 \cdot \frac{z}{z-0{,}8} \longrightarrow \begin{array}{ll} 0{,}41 \cdot 0{,}8^n \cdot u(n) & \text{per } z: |z| > 0{,}8 \\ -0{,}41 \; 0{,}8^n \cdot u(-n-1) & \text{per } z: |z| < 0{,}8 \end{array}$$

$$\frac{0{,}51}{1{,}2}\;\frac{z}{z+\frac{1}{1{,}2}} \longrightarrow \frac{0{,}51}{1{,}2}\cdot\left(-\frac{1}{1{,}2}\right)^n u(n) \quad \text{per } z: |z| > +\frac{1}{1{,}2} \simeq 0{,}83$$

$$-\frac{0{,}51}{1{,}2}\left(-\frac{1}{1{,}2}\right)^n u(-n-1) \quad \text{per } z: |z| < \frac{1}{1{,}2}$$

• valuto ROC



$$ROC_1 = \{z: |z| < 0{,}8\} \longrightarrow x(n) = -0{,}41 \cdot 0{,}8^n\, u(-n-1) - \frac{0{,}51}{1{,}2}\left(-\frac{1}{1{,}2}\right)^n u(-n-1)$$

$$ROC_2 = \left\{z: 0{,}8 < |z| < \frac{1}{1{,}2}\right\} \rightarrow x(n) = 0{,}41 \cdot 0{,}8^n\, u(n) - \frac{0{,}51}{1{,}2}\left(-\frac{1}{1{,}2}\right)^n u(-n-1)$$

$$ROC_3 = \left\{z: |z| > \frac{1}{1{,}2}\right\} \longrightarrow x(n) = 0{,}41 \cdot 0{,}8^n\, u(n) + \frac{0{,}51}{1{,}2}\left(-\frac{1}{1{,}2}\right)^n u(n)$$

• ES)

Trova la trasformata inversa di

$$X(z) = \frac{1-3z^{-5}}{(1-0{,}2z^{-1})(1+0{,}6z^{-1})} \quad \text{per } z: 0{,}2 < |z| < 0{,}6$$

• $N(z)$ ha grado superiore rispetto a quello di $D(z)$! Non posso decomporre in fratti semplici, ma posso scrivere la seguente relazione:

$$X(z) = \frac{1}{(1-0{,}2z^{-1})(1+0{,}6z^{-1})} - 3\cdot z^{-5}\frac{1}{(1-0{,}2z^{-1})(1+0{,}6z^{-1})} = G(z) - 3z^{-5}G(z)$$

$$\updownarrow$$

$$x(n) = g(n) - 3\,g(n-5)$$

• antitrasformo $G(z)$ $\left(\xrightarrow{\quad 0{,}2 \quad 0{,}6 \quad}\right)$

[METODO DEI RESIDUI]

$$g(n) = \frac{1}{2\pi j}\oint_C G(z)z^{n-1}\,dz = \sum_i \text{Res}\left[G(z)z^{n-1},\ z_i = \text{polo interno a } C\right]$$

$$\text{Res}\left[G(z)z^{n-1}, 0{,}2\right] = G(z)\cdot z^{n-1}\cdot(z-0{,}2)\Big|_{z=0{,}2} = \frac{z^{n-1}\cdot z}{\cancel{(z-0{,}2)}(1+0{,}6z^{-1})}\cancel{(z-0{,}2)}\Big|_{z=0{,}2} =$$

$$= z^n\frac{1}{1+0{,}6z^{-1}}\Big|_{z=0{,}2} = (0{,}2)^n\cdot\frac{1}{4}\cdot u(n)$$

$$\text{Res}\left[G(z)\cdot z^{n-1}, -0{,}6\right] = \frac{z^n}{(1-0{,}2z^{-1})(z+0{,}6)}\cancel{(z+0{,}6)}\Big|_{z=-0{,}6} = z^n\cdot\frac{1}{1-0{,}2z^{-1}}\Big|_{z=-0{,}6} =$$

$$= (-0{,}6)^n\frac{1}{1+\frac{1}{3}} = \frac{3}{4}(-0{,}6)^n u(n)$$

$$\Rightarrow g(n) = (0{,}2)^n\cdot\frac{1}{4}u(n) - \frac{3}{4}(-0{,}6)^n u(-n-1) \quad \text{per la ROC scelta}$$

[FRATTI]

$$G(z) = \frac{A_1}{(1-0{,}2z^{-1})} + \frac{A_2}{(1+0{,}6z^{-1})} = \frac{A_1 + 0{,}6A_1z^{-1} + A_2 - 0{,}2A_2z^{-1}}{(\quad)(\quad)} = \frac{N(z)}{D(z)}$$

$$N(z) = z^{-1}(0{,}6A_1 - 0{,}2A_2) + (A_1 + A_2) = 1$$

$$\begin{cases} 0{,}6A_1 - 0{,}2A_2 = 0 \rightarrow A_1 = \frac{1}{3}A_2 \\ A_1 + A_2 = 1 \end{cases} \quad \begin{cases} A_1 = \frac{1}{3}A_2 \\ \frac{4}{3}A_2 = 1 \end{cases} \quad \begin{cases} A_1 = \frac{1}{4} \\ A_2 = \frac{3}{4} \end{cases}$$

$$G(z) = \frac{1}{4}\frac{1}{(1-0{,}2z^{-1})} + \frac{3}{4}\frac{1}{(1+0{,}6z^{-1})} \longrightarrow g(n) = \frac{1}{4}0{,}2^n u(n) - \frac{3}{4}(-0{,}6)^n u(-n-1)$$

• ES)

Trova la trasformata inversa di

$$X(z) = \frac{2 - 4{,}7\, z^{-1} + 2{,}23\, z^{-2} - 0{,}3\, z^{-3}}{1 - 0{,}7\, z^{-1} + 0{,}12\, z^{-2}} \qquad \text{per } z: 0{,}3 < |z| < 0{,}4$$

• il grado del numeratore è > di quello del denominatore. Faccio una divisione tra polinomi per abbassare il grado

$$\begin{array}{cccc|ccc}
-0{,}3\, z^{-3} & 2{,}23\, z^{-2} & -4{,}7\, z^{-1} & 2 & 0{,}12\, z^{-2} & -0{,}7\, z^{-1} & 1 \\
\cline{5-7}
-0{,}3\, z^{-3} & 1{,}75\, z^{-2} & -2{,}5\, z^{-1} & & -2{,}5\, z^{-1} & +4 & \\
\hline
// & 0{,}48\, z^{-2} & -2{,}2\, z^{-1} & 2 & & & \\
 & 0{,}48\, z^{-2} & -2{,}8\, z^{-1} & 4 & & & \\
\cline{2-4}
 & // & 0{,}6\, z^{-1} & -2 & \text{non posso dividere ulteriormente!} & &
\end{array}$$

$$X(z) = \underbrace{-2{,}5\, z^{-1} + 4}_{A(z)} + \underbrace{\frac{0{,}6\, z^{-1} - 2}{1 - 0{,}7\, z^{-1} + 0{,}12\, z^{-2}}}_{G(z)} = A(z) + G(z) \longrightarrow x(n) = a(n) + g(n)$$

• inverto $A(z) \rightarrow a(n) = 4\,\delta(n) - 2{,}5\,\delta(n-1)$

• inverto $G(z) = \dfrac{N(z)}{D(z)}$

poli: $1 - 0{,}7\, z^{-1} + 0{,}12\, z^{-2} = 0 \longrightarrow$ con $K = z^{-1} \longrightarrow 0{,}12\, K^2 - 0{,}7\, K + 1 = 0$

$$K_{1,2} = \frac{0{,}7 \pm \sqrt{0{,}49 - 0{,}48}}{0{,}24} = \frac{0{,}7 \pm 0{,}1}{0{,}24} = \begin{cases} K_1 = \frac{10}{3} \\ K_2 = \frac{10}{4} \end{cases}$$

$\left(1 - \frac{3}{10}K\right)\left(1 - \frac{4}{10}K\right) \longrightarrow$ cambio var. $\longrightarrow \left(1 - 0{,}3\, z^{-1}\right)\left(1 - 0{,}4\, z^{-1}\right)$

$$G(z) = \frac{0{,}6\, z^{-1} - 2}{(1 - 0{,}3\, z^{-1})(1 - 0{,}4\, z^{-1})} = \frac{A_1}{(1 - 0{,}3\, z^{-1})} + \frac{A_2}{(1 - 0{,}4\, z^{-1})} = \frac{A_1 - 0{,}4\, A_1 z^{-1} + A_2 - 0{,}3\, A_2 z^{-1}}{(\quad)(\quad)}$$

$$N(z) = z^{-1}\left(-0{,}4\, A_1 - 0{,}3\, A_2\right) + \left(A_1 + A_2\right) = 0{,}6\, z^{-1} - 2$$

$$\begin{cases} -0{,}4\, A_1 - 0{,}3\, A_2 = 0{,}6 \\ A_1 + A_2 = -2 \end{cases} \qquad \begin{cases} A_1 = \dfrac{0{,}6 + 0{,}3\, A_2}{-0{,}4} = -\dfrac{3}{2} - \dfrac{3}{4} A_2 \\ A_1 + A_2 = -2 \end{cases}$$

$$\begin{cases} \ldots \\ -\frac{3}{2} - \frac{3}{4} A_2 + A_2 = -2 \end{cases} \qquad \begin{cases} \ldots \\ \frac{1}{4} A_2 = -\frac{1}{2} \end{cases} \qquad \begin{cases} A_1 = -\frac{3}{2} - \frac{3}{4}(-2) = 0 \quad !!! \\ A_2 = -2 \end{cases}$$

in effetti $N(z) = 0{,}6\,z^{-1} - 2 \;\; = -2(1 - 0{,}3\,z^{-1})$!!

$$G(z) = \frac{-2}{1 - 0{,}4\,z^{-1}} \longrightarrow g(n) = -2\cdot(0{,}4)^n\, u(n) \quad \text{per } z : |z| > 0{,}4$$

$$g(n) = +2\,(0{,}4)^n\, u(-n-1) \quad \text{per } z : |z| < 0{,}4$$

- metto assieme i risultati:

$$x(n) = a(n) + g(n) = 4\,\delta(n) - 2{,}5\,\delta(n-1) + 2\,(0{,}4)^n\, u(-n-1)$$

• ES 2.7)

Determinare la trasformata inversa di:

$$X(z) = \frac{3z+2}{3z^2-7z+2} = \frac{N(z)}{D(z)}$$

• uso la decomposizione $\frac{X(z)}{z}$

• decompongo il denominatore $D(z)$

$$D(z) = 3z^2 - 7z + 2 = 0 = az^2 + bz + c$$

$$z_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{7 \pm \sqrt{49-24}}{6} = \frac{7 \pm 5}{6} \Rightarrow \begin{matrix} z_1 = 2 \\ z_2 = \frac{1}{3} \end{matrix}$$

$$(z-2)(z-\tfrac{1}{3}) = z^2 - 2z - \tfrac{1}{3}z + \tfrac{2}{3} = z^2 - \tfrac{7}{3}z + \tfrac{2}{3}$$

$$D(z) = 3(z-2)(z-\tfrac{1}{3})$$

• decompongo $\frac{X(z)}{z}$ in fratti semplici

$$\frac{X(z)}{z} = \frac{N(z)}{zD(z)} = \frac{N(z)}{3z(z-2)(z-\frac{1}{3})} = \frac{A_1}{3z} + \frac{A_2}{(z-2)} + \frac{A_3}{(z-\frac{1}{3})} =$$

$$= \frac{A_1(z-2)(z-\frac{1}{3}) + A_2\, 3z(z-\frac{1}{3}) + A_3(3z)(z-2)}{3z(z-2)(z-\frac{1}{3})} = \frac{N(z)}{3z(z-2)(z-\frac{1}{3})}$$

$$N(z) = (A_1 z - 2A_1)(z-\tfrac{1}{3}) + 3A_2 z^2 - A_2 z + 3A_3 z^2 - 6A_3 z =$$

$$= A_1 z^2 - \tfrac{A_1}{3}z - 2A_1 z + \tfrac{2}{3}A_1 + 3A_2 z^2 - A_2 z + 3A_3 z^2 - 6A_3 z =$$

$$= z^2(A_1 + 3A_2 + 3A_3) + z(-\tfrac{7}{3}A_1 - A_2 - 6A_3) + (\quad + \tfrac{2}{3}A_1)$$

$$\begin{cases} A_1 + 3A_2 + 3A_3 = 0 \\ -\frac{7}{3}A_1 - A_2 - 6A_3 = 3 \\ \frac{2}{3}A_1 = 2 \rightarrow A_1 = 3 \end{cases} \quad \begin{cases} 3 + 3A_2 + 3A_3 = 0 \\ -7 - A_2 - 6A_3 = 3 \\ A_1 = 3 \end{cases} \quad \begin{cases} A_2 = -1 - A_3 \\ -7 + 1 + A_3 - 6A_3 = 3 \\ A_1 = 3 \end{cases} \quad \begin{cases} A_2 = -1 - A_3 \\ A_3 = -\frac{9}{5} \\ A_1 = 3 \end{cases}$$

$$\begin{cases} A_2 = -1 + \frac{9}{5} = \frac{4}{5} \\ A_3 = -\frac{9}{5} \\ A_1 = 3 \end{cases}$$

$$\frac{X(z)}{z} = \frac{1}{z} + \frac{4}{5}\frac{1}{(z-2)} - \frac{9}{5}\cdot\frac{1}{(z-\frac{1}{3})}$$

• riscrivo $X(z)$

$$X(z) = z \cdot \frac{X(z)}{z} = 1 + \frac{4}{5} \cdot \frac{z}{z-2} - \frac{9}{5} \frac{z}{z-\frac{1}{3}} = 1 + \frac{4}{5} \frac{1}{1-2z^{-1}} - \frac{9}{5} \frac{1}{1-\frac{1}{3}z^{-1}}$$

• proprietà di linearità

$\left(\text{NOTA: } a^n u(n) \rightarrow \frac{z}{z-a} \quad z: |z| > a \right)$

$$1 \rightarrow \delta(n)$$

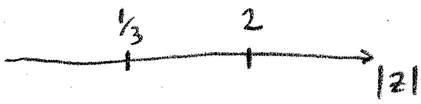
$$\frac{4}{5} \frac{z}{z-2} \rightarrow \frac{4}{5} 2^n u(n) \quad \text{per} \quad z: |z| > 2$$

$$-\frac{4}{5} 2^n u(-n-1) \quad \text{per} \quad z: |z| < 2$$

$$-\frac{9}{5} \frac{z}{z-\frac{1}{3}} \longrightarrow -\frac{9}{5} \cdot \left(\frac{1}{3}\right)^n u(n) \quad \text{per} \quad z: |z| > \frac{1}{3}$$

$$\frac{9}{5} \cdot \left(\frac{1}{3}\right)^n u(-n-1) \quad \text{per} \quad z: |z| < \frac{1}{3}$$

• metto assieme tutto



$$ROC_1 = \left\{ z: |z| < \frac{1}{3} \right\} \rightarrow x(n) = \delta(n) - \frac{4}{5} 2^n u(-n-1) + \frac{9}{5} \left(\frac{1}{3}\right)^n u(-n-1)$$

$$ROC_2 = \left\{ z: \frac{1}{3} < |z| < 2 \right\} \rightarrow x(n) = \delta(n) - \frac{4}{5} 2^n u(-n-1) - \frac{9}{5} \left(\frac{1}{3}\right)^n u(n)$$

$$ROC_3 = \left\{ z: |z| > 2 \right\} \rightarrow x(n) = \delta(n) + \frac{4}{5} 2^n u(n) - \frac{9}{5} \left(\frac{1}{3}\right)^n u(n)$$

• ES)

Calcola l'antitrasformata di

$$X(z)=\frac{z}{2z^3-5z^2+4z-1}$$

• decompongo il denominatore con la regola di Ruffini

prova $z_1=1$:

	2	-5	4	-1
1		2	-3	1
	2	-3	1	0

$\rightarrow (z-1)(2z^2-3z+1) = (z-1)\left(z^2-\frac{3}{2}z+\frac{1}{2}\right)2$

trovo le altre radici

$$z_{2,3}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{\frac{3}{2}\pm\sqrt{\frac{9}{4}-\frac{8}{4}}}{2}=$$

$$z_2=\frac{\frac{3}{2}+\frac{1}{2}}{2}=1$$

$$z_3=\frac{\frac{3}{2}-\frac{1}{2}}{2}=\frac{1}{2}$$

• riscrivo $X(z)$

$$X(z)=\frac{z}{2(z-1)^2(z-\frac{1}{2})}$$

• decompongo in fratti semplici $\frac{X(z)}{z}$

$$\frac{X(z)}{z}=\frac{1}{2(z-1)^2(z-\frac{1}{2})}=\frac{A}{2(z-\frac{1}{2})}+\frac{B}{(z-1)}+\frac{C}{(z-1)^2}=\frac{N(z)}{D(z)}$$

$$N(z)=A(z-1)^2+B\cdot 2\cdot(z-\tfrac{1}{2})(z-1)+C\cdot 2\cdot(z-\tfrac{1}{2})=$$

$$=Az^2+A-2Az+(2Bz-B)(z-1)+2Cz-C=$$

$$=Az^2+A-2Az+2Bz^2-Bz-2Bz+B+2Cz-C=1$$

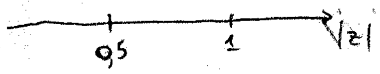
$$\begin{cases} A+2B=0 \\ -2A-B-2B+2C=0 \\ A+B-C=1 \end{cases} \quad \begin{cases} A=-2B \\ 4B-3B+2C=0 \\ -2B+B-1=C \end{cases} \quad \begin{cases} A=-2B \\ \cancel{4B}-3B-\cancel{4B}+2B-2=0 \rightarrow B=-2 \\ \dots \end{cases}$$

$$\begin{cases} A=4 \\ B=-2 \\ C=1 \end{cases}$$

$$\frac{X(z)}{z} = 2\cdot\frac{1}{z-\frac{1}{2}} - 2\cdot\frac{1}{z-1} + \frac{1}{(z-1)^2}$$

$$X(z) = 2\cdot\frac{z}{z-\frac{1}{2}} - 2\cdot\frac{z}{z-1} + \frac{z}{(z-1)^2}$$ ⇒ somma di trasf. note!!

- sfrutto linearità della trasformata



$ROC_1 = \left\{ z : |z| < 0{,}5 \right\} \longrightarrow x(n) = -2\cdot\left(\frac{1}{2}\right)^n u(-n-1) + 2\,u(-n-1) - n\cdot u(-n-1)$

$ROC_2 = \left\{ z : 0{,}5 < |z| < 1 \right\} \longrightarrow x(n) = 2\cdot\left(\frac{1}{2}\right)^n u(n) + 2\,u(-n-1) - n\,u(-n-1)$

$ROC_3 = \left\{ z : |z| > 1 \right\} \longrightarrow x(n) = 2\cdot\left(\frac{1}{2}\right)^n u(n) - 2\,u(n) + n\,u(n)$

con $n\cdot u(n)$ rampa unitaria