- SISTEMI LINEARI TEMPO INVARIANTI

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) * h(n) \iff Y(z) = X(z) \cdot \underline{H(z)}$$

↳ funzione di trasferimento

- TRASFORMATA Z E FOURIER

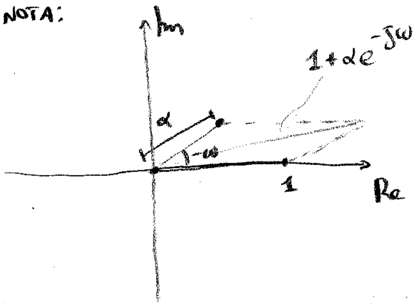
sostituisco $z = e^{j\omega}$, ovvero valuto $H(z)$ sul cerchio unitario

- $\boxed{\text{RELAZIONE GEOMETRICA}}$

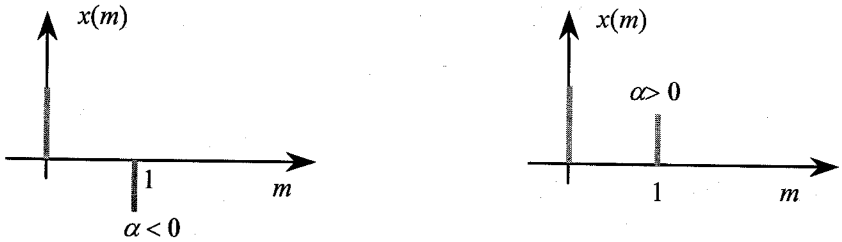
- SINGOLO ZERO

$$H(z) = 1 + \alpha z^{-1}$$
$$H(e^{j\omega}) = 1 + \alpha e^{-j\omega}$$
$$|H(e^{j\omega})| = |1 + \alpha e^{-j\omega}| = |1 + \alpha\cos(\omega) - j\alpha\sin(\omega)| = \sqrt{1 + 2\alpha\cos(\omega) + \alpha^2\cos^2(\omega) + \alpha^2\sin^2(\omega)} =$$
$$= \sqrt{1 + \alpha^2 + 2\alpha\cos(\omega)}$$

NOTA:

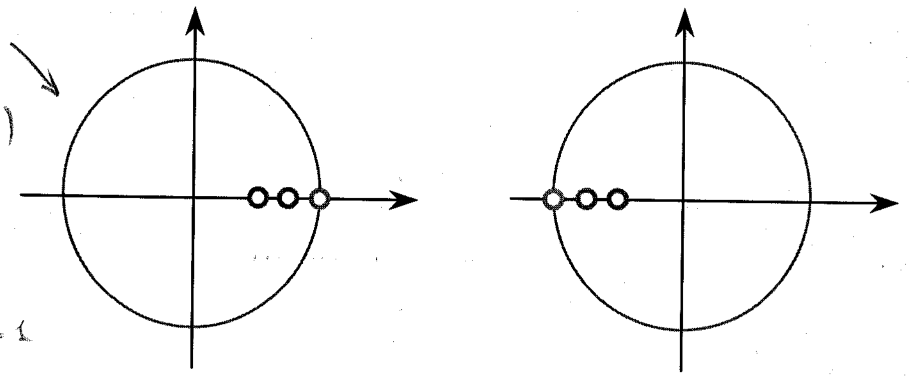


oppure per Eulero → $e^{j\omega} = \cos\omega + j\sin\omega$

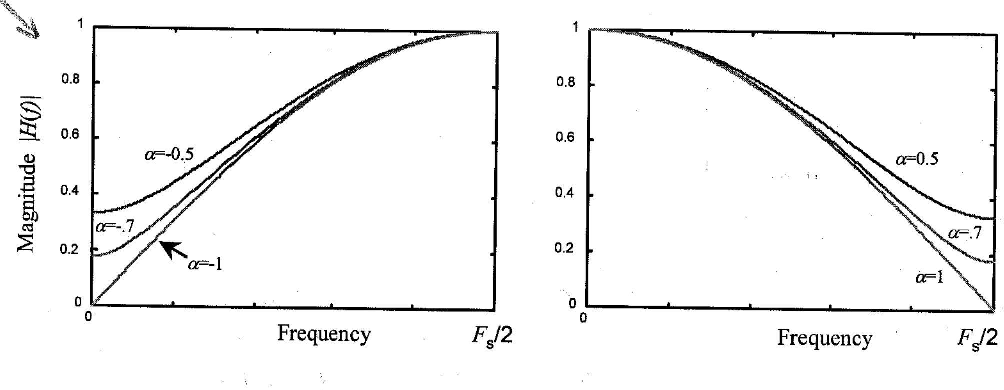


lo zero è in $-\alpha$

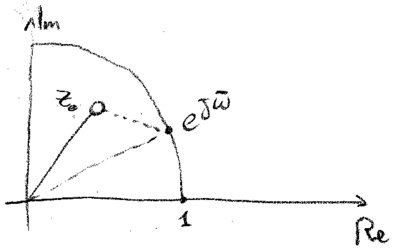
(nota: se lo metto in $-\frac{1}{\alpha}$ non cambio modulo !!)



normalizzate con max = 1



- GENERICO ZERO COMPLESSO (geom.)



$$H(z) = 1 - z_0 z^{-1}$$

$$H(e^{j\bar{\omega}}) = 1 - z_0 e^{-j\bar{\omega}}$$

$$|H(e^{j\bar{\omega}})| = |1 - z_0 e^{-j\bar{\omega}}| = |e^{-j\bar{\omega}}(e^{j\bar{\omega}} - z_0)| = \cancelto{1}{|e^{-j\bar{\omega}}|}\,|e^{j\bar{\omega}} - z_0| :$$

$$= |e^{j\bar{\omega}} - z_0| =$$ distanza tra zero z_0 e punto su circ. con angolo $\bar{\omega}$

- SINGOLO POLO

$$H(z) = \frac{1}{1 + \alpha z^{-1}}$$

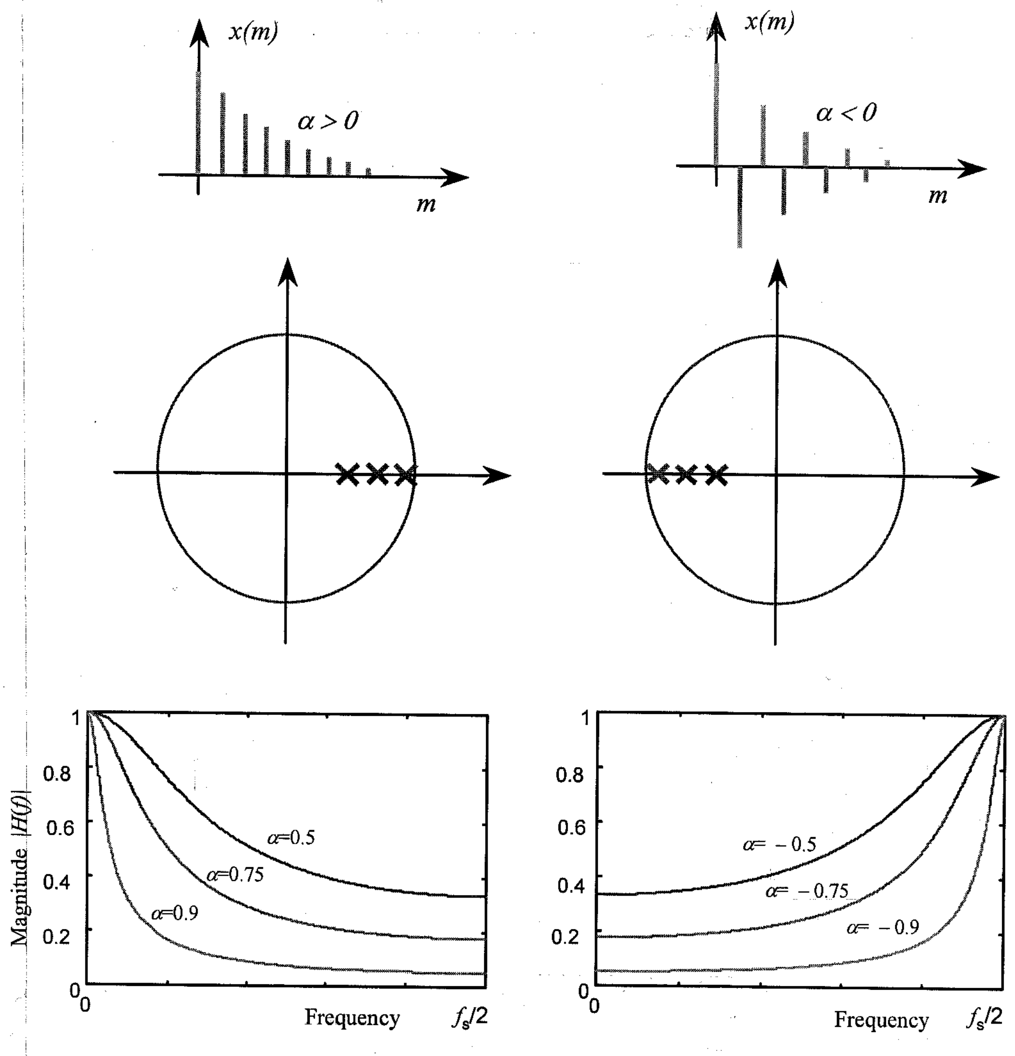
per scrivere la risp. all'impulso devo esplicitare il polinomio in z^{-1}, quindi divido

$$\begin{array}{l|l}
1 & 1 + \alpha z^{-1} \\
\hline
1 + \alpha z^{-1} & 1 - \alpha z^{-1} + \alpha^2 z^{-2} - \alpha^3 z^{-3} \ldots \\
\hline
// \ -\alpha z^{-1} & \\
-\alpha z^{-1} - \alpha^2 z^{-2} & \\
\hline
// \ \ +\alpha^2 z^{-2} & \\
\alpha^2 z^{-2} + \alpha^3 z^{-3} & \\
\hline
// \ \ -\alpha^3 z^{-3} & \\
\ldots &
\end{array}$$

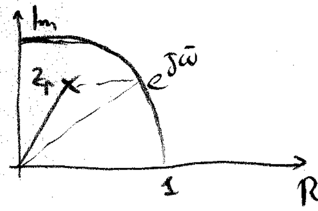
risposta IIR $\left(\sum_{k=0}^{\infty} (-\alpha)^k z^{-k}\right)$

$\Downarrow$

valuto per via geometrica



- GENERICO POLO COMPLESSO (geom)

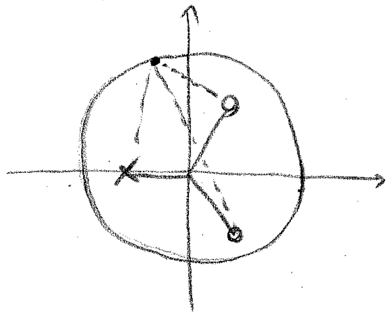


$$H(z) = \frac{1}{1 - z_p z^{-1}}$$

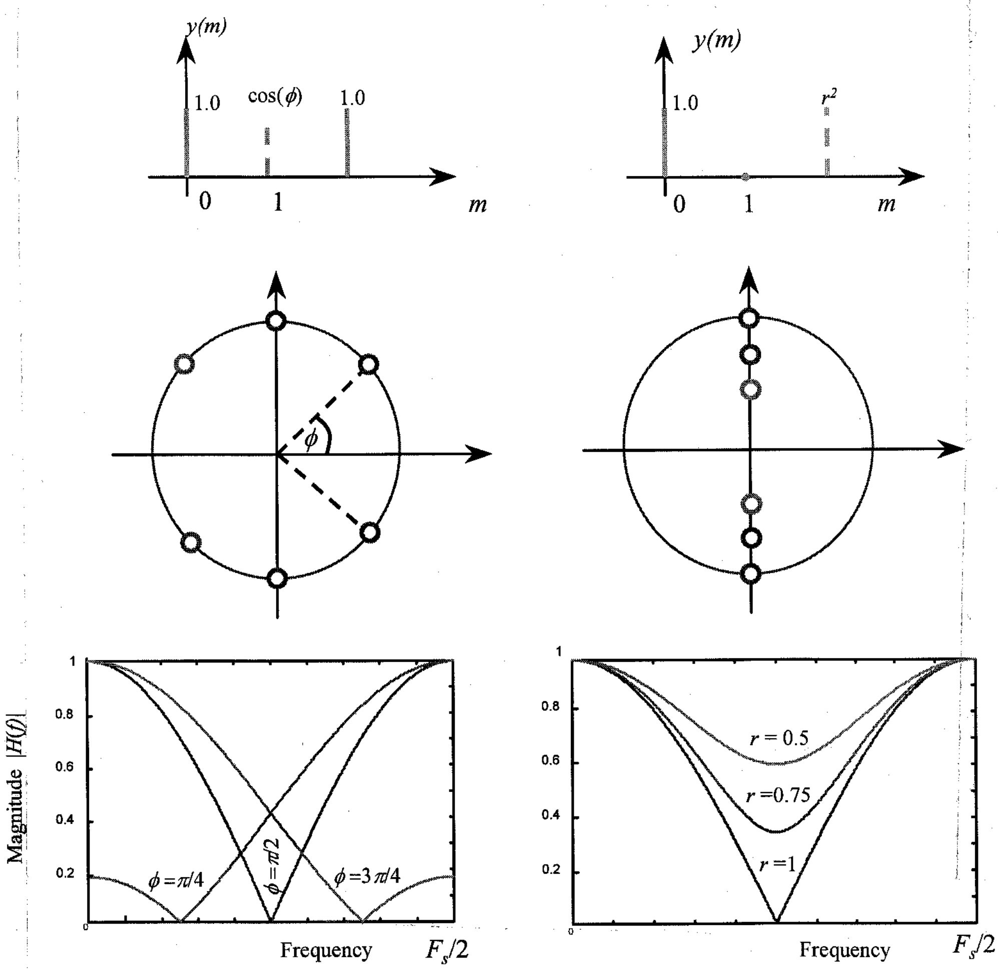
$$|H(e^{j\bar\omega})| = \frac{1}{|1 - z_p e^{-j\bar\omega}|} = \frac{1}{|e^{j\bar\omega} - z_p|}$$

- RELAZIONE GENERALE

$$H(z) = \frac{a_0 \prod_i (1 - z_i z^{-1})}{\prod_j (1 - z_j z^{-1})}$$

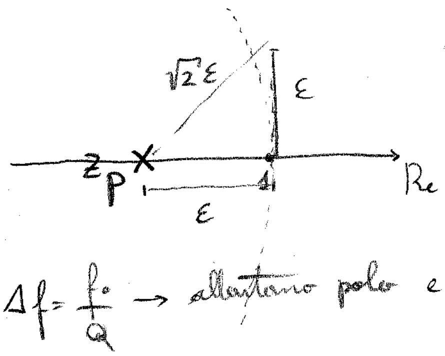


- NOTA: ricorda di mettere poli/zeri reciproci per avere una trasformata simmetrica e quindi segnale reale.



BANDA PASSANTE

- Voglio progettare un passa basso con un polo
- Come controllo la banda passante?



$z_p = 1 - \varepsilon \rightarrow \varepsilon =$ dist. da cerchio unitario

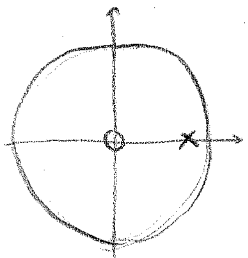
$\varepsilon << 1$

approx $\rightarrow 2\varepsilon =$ banda a 3dB

- $\Delta f = \frac{f_0}{Q} \rightarrow$ allontano polo e Q diminuisce

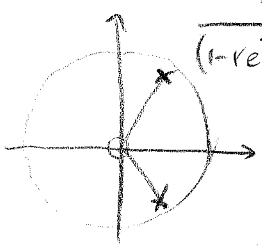
- FILTRI NOTI

P. BASSO $\rightarrow$ 1 polo



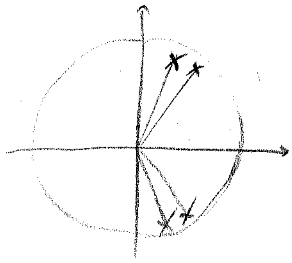
$$\frac{1}{1 - r z^{-1}}$$

P. BANDA $\rightarrow$ 2 poli

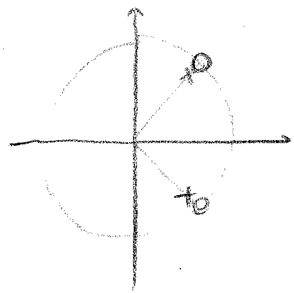


$$\frac{1}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})}$$

o più poli



NOTCH $\rightarrow$ 2 poli + 2 zeri



$$\frac{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}{(1 - r e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}$$

i poli compensano l'effetto degli zeri per $\omega \neq \omega_0$

(vedere in matlab con fvtool e zplane)

• ES)

Ho un segnale campionato a $f_s = 1000$ Hz; 1) Dimensiona un filtro p. basso con banda a 3 dB di ± 25 Hz. 2) modificare il filtro per ottenere un p. banda con $f_0 = 125$ Hz

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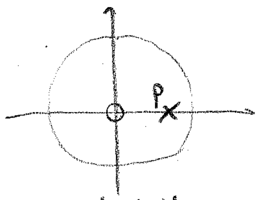
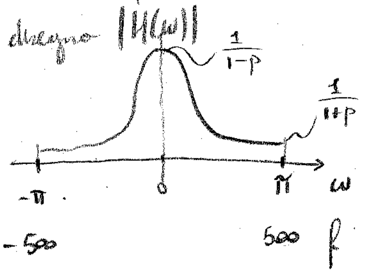
1). p. basso $\rightarrow$ 1 polo reale $\rightarrow$

$$H(z) = \frac{1}{1 - p z^{-1}}$$



$$H(\omega) = \frac{1}{1 - p e^{-j\omega}}$$

• disegno $|H(\omega)|$



$$\omega = 0 \rightarrow |H(0)| = \frac{1}{|1-p|} = \frac{1}{1-p}$$ (il polo è nel cerchio)

$$|H(\pi)| = \frac{1}{|1+p|} = \frac{1}{1+p}$$

• trovo $\Delta\omega$

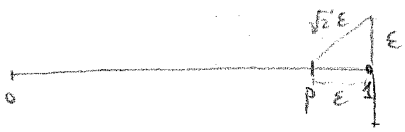
$$\Delta f = 50 \text{ Hz} \rightarrow \frac{\Delta f}{f_s} = \frac{\Delta\omega}{2\pi} \rightarrow \Delta\omega = 2\pi \cdot \frac{50}{1000} = \frac{\pi}{10}$$

• impongo vincolo 3 dB

$$\frac{\left|H\left(\pm\frac{\Delta\omega}{2}\right)\right|^2}{|H(0)|^2} = \frac{1}{2} \Rightarrow \left|\frac{1}{1 - p e^{-j\frac{\pi}{20}}}\right| = \frac{1}{\sqrt{2}}\left|\frac{1}{1-p}\right|$$ posso risolvere in p

• per non risolvere in p uso appr. vettoriale

$p = 1 - \varepsilon$ con $\varepsilon \ll 1$

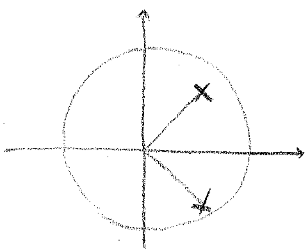


$$\Delta\omega = 2\varepsilon \Rightarrow \varepsilon = \frac{\Delta\omega}{2} = \frac{\pi}{20} \Rightarrow p = 1 - \frac{\pi}{20} \simeq 0{,}84$$

2). • trovo ω_0 corrispondente a f_0

$$\frac{\omega_0}{2\pi} = \frac{f_0}{f_s} \rightarrow \omega_0 = 2\pi \cdot \frac{125}{1000} = \frac{\pi}{4}$$

• per p. banda uso coppia di poli (filtro reale)



$$H(z) = \frac{1}{\left(1 - p e^{j\frac{\pi}{4}} z^{-1}\right)\left(1 - p e^{-j\frac{\pi}{4}} z^{-1}\right)}$$

ES)

mostrare che $H(z) = \dfrac{1 - z^{-16}}{1 - 2\cos(\pi/8) z^{-1} + z^{-2}}$ ha una risposta di tipo FIR

- per essere FIR, i poli si devono eliminare. Calcolo i "poli"

$1 - 2\cos\left(\frac{\pi}{8}\right) z^{-1} + z^{-2} = 0$

$$z^{-1}_{1,2} = \frac{2\cos \pm \sqrt{4\cos^2 - 4}}{2} = \frac{2\cos \pm \sqrt{4(\cos^2 - 1)}}{2} = \frac{2\cos \pm 2\sqrt{-(1-\cos^2)}}{2} = \frac{2\cos\left(\frac{\pi}{8}\right) \pm 2 j \sin\left(\frac{\pi}{8}\right)}{2} =$$

$$= \cos\left(\frac{\pi}{8}\right) \pm j \sin\left(\frac{\pi}{8}\right)$$

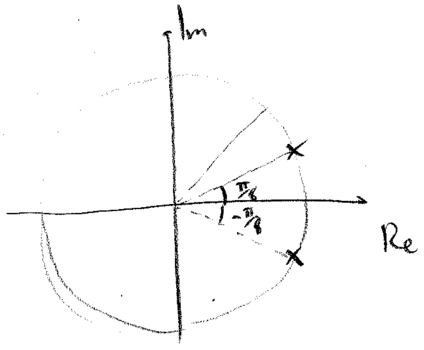
$\rho_{1,2} = \sqrt{\cos^2\left(\frac{\pi}{8}\right) + \sin^2\left(\frac{\pi}{8}\right)} = 1$

$\omega_{1,2} = tg^{-1}\left(\frac{\pm \sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}\right) = \pm\frac{\pi}{8}$

$\Rightarrow$ poli con modulo 1 e fase $\pm\frac{\pi}{8}$ $\left(1 \cdot e^{\pm j\frac{\pi}{8}}\right)$

$\Downarrow$

posso scrivere il denom come $\left(1 - e^{j\frac{\pi}{8}} z^{-1}\right)\left(1 - e^{-j\frac{\pi}{8}} z^{-1}\right)$



- Calcolo gli zeri:

a) - ho 16 zeri
- $z = 1$ è uno zero

} gli zeri partono da 1 e si sfasano di $\frac{2\pi}{16} = \frac{\pi}{8}$ per avere molteplicità 16

b) $z^{16} = 1 \rightarrow (\rho \cdot e^{j\omega})^{16} = 1 \rightarrow \rho^{16} \cdot e^{j\omega 16} = 1 = 1 \cdot e^{jK2\pi} \Rightarrow \begin{cases} \rho^{16} = 1 \\ \omega 16 = 2\pi K \end{cases} \begin{cases} \rho = 1 \\ \omega = K \cdot \frac{\pi}{8} \end{cases}$

- 2 poli sono cancellati da 2 zeri !!

- ES) e se $H(z) = 1 + z^{-16}$?

- gli zeri sono gli z che alla 16 danno $-1 \rightarrow z^{16} = -1$

$z^{16} = -1 \rightarrow (\rho e^{j\omega})^{16} = -1 \rightarrow \rho^{16} \cdot e^{j\omega 16} = -1 \rightarrow \rho^{16} e^{j\omega 16} = 1 \cdot e^{j(2K+1)\pi}$

tutti i multipli dispari di π (i.e. $\pi, 3\pi, 5\pi, \ldots$

$\begin{cases} \rho^{16} = 1 \\ \omega 16 = (2K+1)\pi \end{cases} \begin{cases} \rho = 1 \\ \omega = (2K+1)\frac{\pi}{16} \end{cases}$ (sono spaziati di $\frac{\pi}{8}$)

$\omega_0 = \frac{\pi}{16}$

$\omega_1 = 3\frac{\pi}{16}$

$\vdots$

$\omega_{15} = 31\frac{\pi}{16}$ e poi ricomincia.

• ES)

Controlla l'errore su Δw dell'es precedente

$$H(0) = \frac{1}{1-p}$$

$$H\left(\frac{\Delta w}{2}\right) = H\left(\frac{\pi}{20}\right) = \frac{1}{1-pe^{-j\frac{\pi}{20}}}$$

$$p = 1 - \frac{\pi}{20}$$

vorrei che $\left|H\left(\frac{\Delta w}{2}\right)\right| \simeq \frac{1}{\sqrt{2}} \left|H(0)\right|$

$$|H(0)| = \frac{1}{|1-p|} \simeq 6{,}36$$

$$\left|H\left(\frac{\pi}{20}\right)\right| = \frac{1}{\left|1-pe^{-j\frac{\pi}{20}}\right|} = \frac{1}{\sqrt{1+p^2-2p\cos}} = \frac{1}{0{,}2131} \simeq 4{,}69$$

$$\hookrightarrow \left|1 - p\left(\cos\left(\frac{\pi}{20}\right) - j\sin\left(\frac{\pi}{20}\right)\right)\right| = \sqrt{1+p^2\cos^2 - 2p\cos + \sin^2 p^2} = \sqrt{1+p^2-2p\cos}$$

$$\boxed{\begin{array}{l} \dfrac{6{,}36}{4{,}69} = 1{,}3561 \\ \\ \sqrt{2} = 1{,}4142 \end{array}}$$

$\simeq 4\%$ di errore